Truncated eigenvalue equation and long wavelength behavior of lattice gauge theory

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We review our new method, which might be the most direct and efficient way for approaching the continuum physics from Hamiltonian lattice gauge theory. It consists of solving the eigenvalue equation with a truncation scheme preserving the continuum limit. The efficiency has been confirmed by the observations of the scaling behaviors for the long wavelength vacuum wave functions and mass gaps in (2+1)-dimensional models and (1+1)-dimensional σ model even at very low truncation orders. Most of these results show rapid convergence to the available Monte Carlo data, ensuring the reliability of our method.

1. INTRODUCTION

The main purpose of our work is to approach the scaling region and extract physical results by analytic calculations. For a lattice calculation, a basic requirement is that for weak enough coupling, the dimensionless quantities should satisfy the scaling law, predicted by renormalization group equation. For $SU(N_c)$ gauge theories in 3 dimensions, superrenormalizabilty and dimensional analysis tell us that the dimensionless masses aM should scale as

$$\frac{aM}{g^2} \to \frac{M}{e^2}. \tag{1}$$

For (2+1)-dimensional compact U(1) and (3+1)-dimensional non-abelian gauge theories, aM should scale exponentially as

$$aM \to exp(-b/g^2). \tag{2}$$

If the calculated M data converge to a stable value, we can get an estimate for the mass.

There have been various analytic methods available in the literature (for a review see [1]). The main difficulty of the conventional methods (e.g. strong coupling expansion) is that they converge very slowly and very higher order $1/g^2$ cal-culations are required to extend the results to the intermediate coupling region. Unfortunately, high order calculations are difficult in practice.

Recently, we proposed a new method [2–5] for Hamiltonian lattice gauge theory. This method consists of solving the eigenvalue equation with a suitable truncation scheme preserving the continuum limit. Even at low order truncation, clear scaling windows for the physical quantities in most cases have been established, and the results are in perfect agreement with the available Monte Carlo data. Here we review only the work on $U(1)_3$, $SU(2)_3$ and 2 dimensional σ model, while that for $SU(3)$ has been summarized in [6].

2. THE METHOD

The Schrödinger equation $H|\Omega\rangle = \epsilon_\Omega|\Omega\rangle$ on the Hamiltonian lattice for the ground state

$$|\Omega\rangle = exp[R(U)]|0\rangle \tag{3}$$

and vacuum energy ϵ_Ω can be reformulated as

$$\sum_l \{[E_l, [E_l, R(U)]] + [E_l, R(U)][E_l, R(U)]\}$$

$$-\frac{2}{g^4}\sum_p tr(U_p + U_p^\dagger) = \frac{2a}{g^2}\epsilon_\Omega. \tag{4}$$

To solve this equation, let us write $R(U)$ in order of graphs $G_{n,i}$, i.e., $R(U) = \sum_n R_n(U) = \sum_{n,i} C_{n,i} G_{n,i}(U)$. Substituting it to (4), we have the Nth order truncated eigenvalue equation

$$\sum_l \{[E_l, [E_l, \sum_n^N R_n(U)]]$$

$$+ \sum_{n_1+n_2 \leq N} [E_l, R_{n_1}(U)][E_l, R_{n_2}(U)]\}$$

$$- \frac{2}{g^4} \sum_p tr(U_p + U_p^\dagger) = \frac{2a}{g^2}\epsilon_\Omega. \tag{5}$$

By taking the coefficients of the graphs $G_{n,i}$ in this equation to zero, we obtain a set of nonlinear algebra equations, from which $C_{n,i}$ are determined. The similar method applies to the eigenvalue equation for the mass and its wave function [5]. Therefore, solving lattice field theory is reduced to solving the algebra equations.

The lowest order graph is quite simple: $R_1(U) = C_{1,1}(U_p + h.c.)$ The first term in (5) doesn't generate new graphs, but the second term does, i.e.

$$[E_l, G_{n_1}(U)] \in R_n(U) + lower\ orders,$$

$$[E_l, G_{n_1}(U)][E_l, G_{n_2}(U)] \in R_{n_1+n_2}(U)$$

$$+ lower\ orders. \tag{6}$$

Two questions arise:
1) Should all the new graphs generated by the second term in (5) be taken as independent graphs of order $n_1 + n_2$? For abelian gauge theories, the answer is yes. For non-abelian gauge theories, because of the uni-modular conditions [4–6,13], there is a mixing problem not only for the graphs of the same order, but also for graphs of different orders. The classification for independent graphs is particularly complecate for SU(3).
2) For $n_1 + n_2 > N$, should we keep the lower order graphs in $[E_l, G_{n_1}(U)][E_l, G_{n_2}(U)]$? To preserve the correct limit, at Nth order truncation, one should to DROP all these graphs. This is the essential feature of our truncation scheme, which differs sufficiently from the scheme in [7].

There have also been some other truncation schemes proposed in [8,9]. One of their major problems is the violation of the long wavelength structure or continuum limit of the equation, and consequently the violation of the scaling law (1) or (2) for the physical quantities.

Let's see further why the equation (5) should be truncated in the way suggested at point 2). The continuum limit of a graph $G_{n,i}(U)$ is

$$G_{n,i}(U) = e^2 a^4 [A_{n,i}\ tr(\mathcal{F}^2)$$

$$+ a^2 B_{n,i}\ tr(\mathcal{DF})^2 + ...] \tag{7}$$

with $\mathcal{F}$ the field strength tensor and $\mathcal{D}$ the covariant derivative. It has been generally proven [2] that in the continuum limit the second term of (5) term should behave as

$$[E_l, G_{n_1}(U)][E_l, G_{n_2}(U)] \propto e^2 a^6\ Tr(\mathcal{DF}_{\mu,\nu})^2. \tag{8}$$

To preserve this correct limit, when the equation (5) is truncated to the Nth order, all the graphs created by $[E_l, R_{n_1}(U)][E_l, R_{n_2}(U)]$ for $n_1 + n_2 \leq N$ must be considered. On the other hand, all the graphs created by this term for $n_1 + n_2 > N$ should be dropped, even there are lower order graphs. Otherwise the partial sum of the lower order graphs would make this term behave in a considerably different (wrong) way.

3. RESULTS

Once the coefficients $C_{n,i}$ are obtained by solving (5), we can use (3) and (7) to compute the parameters μ_0 and μ_2 in the vacuum wave function for the long wavelength configurations U [10]

$$|\Omega\rangle = exp[-\mu_0 \int d^{D-1}x\ tr\mathcal{F}^2$$

$$- \mu_2 \int d^{D-1}x\ tr(\mathcal{DF})^2]. \tag{9}$$

The results for μ_0 and μ_2 in 3 dimensional SU(2) gauge theory are shown in Fig. 1. Empressively, nice scaling behavior is obtained even at $N = 3$, and the data for $4/g^2 > 4$ are in good agreement with the Monte Carlo measurements [10]. The order $N = 4$ data are also included in
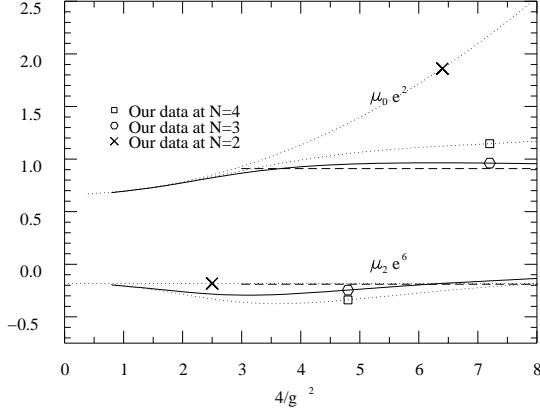
Figure 1. Parameters in the vacuum wave function of $SU(2)_3$ model. The dashed lines show the mean values for the Monte Carlo data.

Figure 2. The mass gap of the $SU(2)_3$ model from three different methods. The dashed lines show the continuum limit of the Monte Carlo data.

this figure. Although there are no big differences between the results at $N = 3$ and $N = 4$, higher order calculation seems necessary to ensure that the results would finally occur at the correct values.

Figure 2 shows our results for the mass gap in $SU(2)_3$ at different truncation orders. For comparison, the results from the truncation method of Llewellyn Smith and Watson (LS-W) [8] and $14th$ order series expansion [11] are also included. Again, even at $N = 3$, with satisfaction of the scaling law, our results are in best agreement with the data taken from the continuum limit of the Monte Carlo data [12].

As is mentioned above, for non-abelian gauge theories, the uni-modular conditions lead to the existence of different choices for independent graphs. In $SU(2)$, because of $trU_p^\dagger = trU_p$, all the disconnected graphs can be transformed into the connected ones, used as an independent set of the graphs. Our results in Figs. 1 and 2 are from such a choice, while the comparison between different choices (connected, disconnected [2] and inverse) has been made in [13]. Of course, one is free to choose arbitrary set of independent graphs. A criterion for a good choice is that it is convergent more rapidly to the continuum limit than other ones at lower order truncation. This has been obviously demonstrated for $SU(3)$ [6,14].
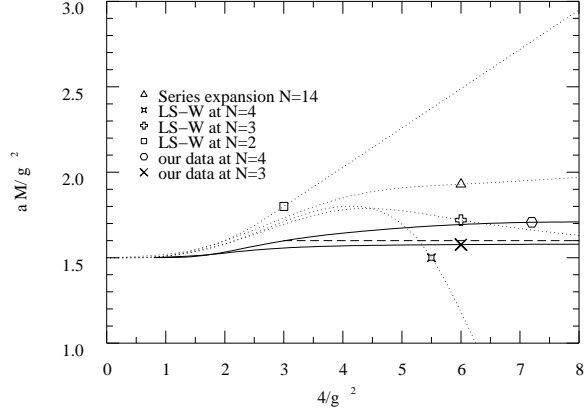
The most intriguing scaling law is the exponential scaling (2). Before investigating QCD in 3+1 dimensions, we would like to test our method in a (2+1)-dimensional compact $U(1)$ model, which has many properties of the realistic theory. Here there is no ambiguity induced by uni-modular conditions in $SU(N_c)$ theories. Because the nature of the abelian group greatly simplifies the calculations, we can easily write a program at arbitary orders.
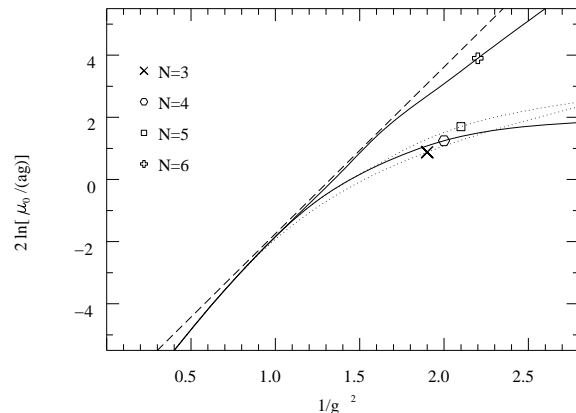

Figure 3. Relevant quantity for μ_0 in the long wavelength vacuum state of compact $U(1)_3$. The dashed line is the expected scaling law.
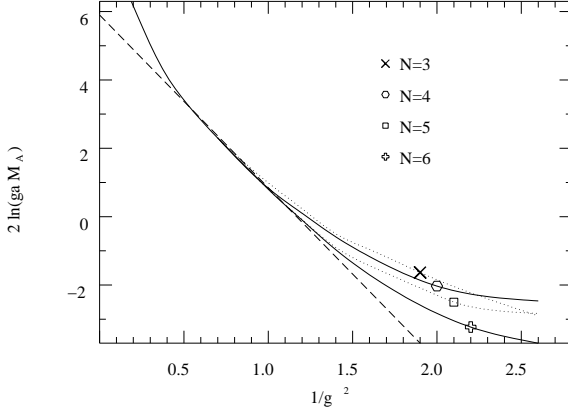
Figure 4. Relevant quantity for the asymmetric mass M_A of compact $U(1)_3$. The dashed line is the expected scaling law.

The results for $U(1)_3$ are shown in Figs. 3 and 4 respectively. At relatively low truncation orders (comparing to 16th order series expansion [11]), a scaling window has been seen around $1/g^2 = 1$, and such a window becomes wider with the order N. Most impressively, there is an obvious tendency to converge to the scaling curve (dash line). This implies that the obtained physical quantities occur at their correct values. Fitting the N=6 data in the scaling region, we get [15]

$$\frac{\mu_0}{ag} = 3.1120 \times 10^{-2} exp(2.54(3)/g^2),$$

$$(M_A ag)^2 = 365(73) exp(-5.0(2)/g^2), \qquad (10)$$

or $\mu_0 M_A = 0.59(5)$.

The non-linear (1+1)-dimensional σ model is another interesting application of our method. According to the theoretical expectation,

$$Ma \propto \frac{1}{g^2} exp(-\frac{2\pi}{g^2} - \frac{g^2}{8\pi}). \qquad (11)$$

Calculations at $N = 5, 6, 7, 8$ have been carried out. Preliminary results are quite encouraging, and they are in reasonable agreement with the Monte Carlo data [16]. To reach the asymptotic scaling region, higher order calculation seems necessary.

4. SUMMARY

The reason for the success of our method is that the truncated eigenvalue equation preserves the continuum limit. The results for (2+1)-dimensional models and (1+1)-dimensional σ models are presented to support the efficiency and reliability of our method. In conclusion, the eigenvalue equation with a proper truncation scheme may be the most direct and efficient way for extracting the continuum physics.

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